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The role of power law nonlinearity in the discrete nonlinear Schrödinger equation on the formation of stationary localized states in the Cayley tree

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Abstract. We study the formation of stationary localized states using the discrete nonlinear Schrödinger equation in a Cayley tree with connectivity K. Two cases, namely, a dimeric power law nonlinear impurity and a fully nonlinear system are considered. We introduce a transformation which reduces the Cayley tree into an one dimensional chain with a bond defect. The hopping matrix element between the impurity sites is reduced by $1/\sqrt{K}$. The transformed system is also shown to yield tight binding Green's function of the Cayley tree. The dimeric ansatz is used to find the reduced Hamiltonian of the system. Stationary localized states are found from the fixed point equations of the Hamiltonian of the reduced dynamical system. We discuss the existence of different kinds of localized states. We have also analyzed the formation of localized states in one dimensional system with a bond defect and nonlinearity which does not correspond to a Cayley tree. Stability of the states is discussed and stability diagram is presented for few cases. In all cases the total phase diagram for localized states have been presented.

PACS. 71.55.-i Impurity and defect levels – 72.10.Fk Scattering by point defects, dislocations, surfaces, and other imperfections (including Kondo effect)

1 Introduction

One well studied nonlinear equation in condensed matter physics and optics is the discrete nonlinear Schrödinger equation (DNLSE). The DNLSE is a nonintegrable standard discretization of the integrable nonlinear Schrödinger equation [1]. The DNLSE in one dimension in its general form is a set of n coupled nonlinear differential equations.

$$i\frac{dC_m}{dt} = -\chi_m f_m(|C_m|)C_m + V_{m,m+1}C_{m+1} + V_{m,m-1}C_{m-1}$$
$$\text{where} \quad V_{m,m+1} = V^{\star}_{m+1,m}; \text{ and } m = 1, 2, 3, ...n. \tag{1}$$

In equation (1) the nonlinearity appears through functions $f_m(|C_m|)$ and χ_m is the nonlinearity parameter associated with the m-th grid point. Since, $\sum_m |C_m|^2$ is made unity by choosing appropriate initial conditions, $|C_m|^2$ can be interpreted as the probability of finding a particle at the m-th grid point. The analytical solutions of equation (1) in general are not known. Numerous works, both analytical as well as numerical, on the DNLSE have been reported [1–16]. As for its application particularly in condensed matter physics, we cite among others the exciton propagation in Holstein molecular crystal chain [2]. In general, the exciton propagation in quasi one dimensional systems [17] having short range electron phonon interaction can be adequately modeled by the DNLSE. Other examples include nonlinear optical responses in superlattices formed by dielectric or magnetic slabs [18] and the mean field theory of a periodic array of twinning planes in the high T_c superconductors [19].

One important feature of the DNLSE is that this can yield stationary localized (SL) states. This is intricately related to the discretization and the consequent nonintegrability of the DNLSE [20]. To understand this we note that the continuous nonlinear Schrödinger equation is integrable and it yields soliton, multisoliton and multisoliton bound states [21]. The soliton solution of the DNLSE has also been investigated by peturbative method [22,23]. The starting point of the approach is the Ablowitz-Ladik equation [24] which is discrete but integrable. It has been shown that from the DNLSE soliton in the form of a kink can be obtained under restrictive conditions. But mostly the solution has center of mass of the soliton executing oscillatory motion [23]. This is due to the nonintegrability of the DNLSE. Then in the limit of small oscillation we basically obtain SL states. In other words SL states are low energy excitations in the system described by the DNLSE in the quasicontinuum limit. On the other hand, stronger discretization destroys moving soliton altogether [23]. Hence SL states are the most prominent solutions of the equation. In the presence of gliding forces like applied

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electric field the SL state in the quasicontinuum limit can go to the finite oscillation state and finally to the moving soliton phase. Hence the study of SL states is important for understanding transport properties of conducting polymers like trans-polyactylene. These SL states might also play a relevant role in the nonlinear DNA dynamics [25] and in the energy localization in nonlinear lattices [26].

As discussed previously, systems described by the DNLSE can yield SL states. It is also well known that in one dimensional systems these states will fall exponentially [27,28] asymptotically. So, to understand the formation of SL states in fully nonlinear systems, it is necessary to consider SL states due to nonlinear clusters embedded in an otherwise perfect lattice. Outside the cluster these states will fall exponentially for one dimensional lattices [29]. Various SL states can arise then due to nodes in the states inside the cluster. Again the energy of the state increases with the increase of nodes. In large clusters one can obtain states with large number of nodes. The energy of these states will also be large. On the other hand, most dominant ones have low energy and consequently less number of nodes. Hence it is sufficient to consider small clusters. This is the motivation for the study of nonlinear dimer embedded in a perfect lattice.

It has been shown that the presence of a nonlinear impurity can produce SL states in one, two and three dimension [20,27–34]. The formation of SL states due to the presence of a single and a dimeric nonlinear impurity in few linear hosts has been studied in details [34]. The same problem is also studied starting from an appropriate Hamiltonian [29]. The fixed point of the Hamiltonian [20,27–29] which generates the appropriate DNLSE can also produce the correct equations governing the formation of SL states. We further note that the appropriate ansatz for the dimer problem has been obtained in our earlier analysis [34]. Furthermore, the formation of intersite peaked and dipped stationary localized states has been studied using the dimeric ansatz. The effect of one nonlinear impurity as well as a dimeric impurity in an otherwise perfect nonlinear chain on the formation of SL states has also been studied [28]. The other important aspect to be considered is the effect of connectivity on the formation of SL states. The simplest way to study this is to consider the Cayley tree. The effect of one power law nonlinear impurity in an otherwise perfect Cayley tree has already been considered [34]. So, we plan here to study the effect of a dimeric nonlinear impurity in an otherwise perfect Cayley tree. We further consider the fully nonlinear Cayley tree. For this purpose a transformation is devised to map the system to an effective one dimensional chain with a bond defect. Consequently, from the symmetry consideration, the dimeric ansatz is found to be suitable for this study. The appropriate form of the monomeric ansatz is also found for this case to study the formation of on-site peaked solitons.

The organization of the paper is as follows. In Section 2 we introduce a transformation which reduces the Cayley tree into a one dimensional system with a bond defect. This transformation has also been checked through Green's function analysis in Appendix A. SL states in the Cayley tree in presence of a dimeric nonlinear impurity is discussed in Section 3. In Section 4, fully nonlinear Cayley tree is discussed. Finally in Section 5 we summarize our investigations. Section 6 contains Appendix A and Appendix B.

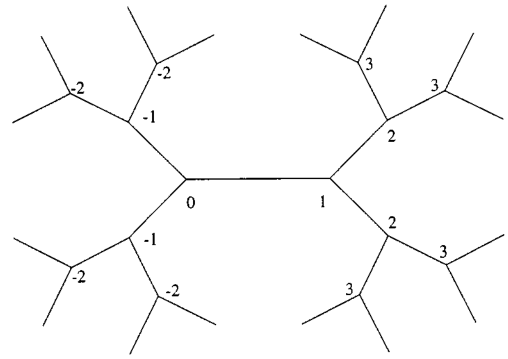

Fig. 1. Cayley tree with connectivity 2. All bonds are of same length.

2 Transformation of Cayley tree to one dimensional system

The structure of a Cayley tree with connectivity, $K = Z - 1$ is shown in Figure 1. Z is the coordination number. We pick up a connection and its two ends are numbered 0 and 1 respectively without any loss of generality. Furthermore, all points in a given generation lie in a shell. Shells are number by n and $n \in Z$ as shown in Figure 1. In a perfect Cayley tree the number of points in the $n^{\rm th}$ shell is K^{n-1} if $n \geq 1$ and $K^{|n|}$ if $n \leq 0$. We further note that for a perfect Cayley tree all points in a given shell have identical neighborhood.

We consider now the motion of a particle on a Cayley tree with connectivity, K. In the tight binding formalism with nearest neighbor hopping only equations governing the motion of the particle are

$$\begin{aligned}
i\frac{d\widetilde{C}_n}{dt} &= K\widetilde{C}_{n+1} + \widetilde{C}_{n-1} + \widetilde{\epsilon}_n\widetilde{C}_n, && n > 1\\
i\frac{d\widetilde{C}_n}{dt} &= K\widetilde{C}_{-|n|-1} + \widetilde{C}_{-|n|+1} + \widetilde{\epsilon}_n\widetilde{C}_n, && n < 0\\
i\frac{d\widetilde{C}_1}{dt} &= K\widetilde{C}_2 + \widetilde{C}_0 + \widetilde{\epsilon}_1\widetilde{C}_1\\
i\frac{d\widetilde{C}_0}{dt} &= K\widetilde{C}_{-1} + \widetilde{C}_1 + \widetilde{\epsilon}_0\widetilde{C}_0. && (2)
\end{aligned}$$

Here $\widetilde{C}_n$ denotes the probability amplitude at any point in the $n^{\rm th}$ shell and all points in the $n^{\rm th}$ cell have the same probability amplitude because of identical neighborhood. The nearest neighbor hopping matrix, V has been taken to be unity without any loss of generality. It is also assumed that all points in a given shell arising due to a specific organization have same site energy. We, however, note that in our subsequent work with the DNLSE this assumption will be automatically satisfied. The normalization condition for the site amplitudes gives

$$\sum_{-\infty}^{0} K^{|n|}|\widetilde{C}_n|^2 + \frac{1}{K}\sum_{n=1}^{\infty} K^n|\widetilde{C}_n|^2 = 1. \tag{3}$$

We now make the following transformations. (i) $\tau = \sqrt{K}t$, (ii) $\epsilon_n = \widetilde{\epsilon}_n/\sqrt{K}$, (iii) $\widetilde{C}_n = K^{-(n-1)/2}C_n$, for $n \geq 1$ and (iv) $\widetilde{C}_{-|n|} = K^{-|n|/2}C_n$ for $n \leq 0$. After substituting these transformations in equation (2) we finally obtain

$$\begin{aligned}
i\frac{dC_n}{d\tau} &= C_{n+1} + C_{n-1} + \epsilon_n C_n, \quad \text{for} \quad n > 1 \quad \text{and} \quad n < 0.\\
i\frac{dC_1}{d\tau} &= C_2 + \frac{1}{\sqrt{K}}C_0 + \epsilon_1 C_1,\\
i\frac{dC_0}{d\tau} &= C_{-1} + \frac{1}{\sqrt{K}}C_1 + \epsilon_0 C_0.
\end{aligned} \tag{4}$$

Furthermore from equation (3) normalization condition reduces to $\sum_{-\infty}^{\infty}|C_n|^2 = 1$. So, the motion of a particle on a Cayley tree is mapped to that on a one dimensional chain. However, in this chain the nearest neighbor hopping matrix element between the zeroth and first site is reduced from unity to $1/\sqrt{K}$. In Appendix A we show that the Green's function $G_{0,0}(E)$ calculated from equation (4) will yield the $\widetilde{G}_{0,0}(\widetilde{E} = E\sqrt{K})$ of a Cayley tree with connectivity K.

Since we are interested in the DNLSE with general power law nonlinear impurity, in our case $\epsilon_n = \widetilde{\epsilon}_n/\sqrt{K} = \widetilde{\chi}_n K^{-(n-1)\sigma/2}K^{-1/2}|C_n|^\sigma$ for $n \geq 1$ and $\epsilon_{-|n|} = \widetilde{\epsilon}_{-|n|}/\sqrt{K} = \widetilde{\chi}_{-|n|}K^{-|n|\sigma/2}K^{-1/2}|C_{-|n|}|^\sigma$ for $n \geq 0$. Furthermore, $\chi_n = \widetilde{\chi}_n/\sqrt{K}$. We further note that a model derivation of power law nonlinearity is given in reference [34]. When all points have the same nonlinearity strength, we have $\chi_n = \chi$, $n \in Z$. On the other hand for a dimeric nonlinear impurity, $\chi_n = \chi(\delta_{n,0} + \delta_{n,1}), n \in Z$. Furthermore, the classical Hamiltonian which can generate equation (4) is

$$\begin{aligned}
H =& 2\sum_{n=-\infty}^{0} \frac{\chi_n}{\sigma+2}V^{|n|\sigma}|C_n|^{\sigma+2}\\
&+ 2\sum_{n=1}^{\infty}\frac{\chi_n}{\sigma+2}V^{(n-1)\sigma}|C_n|^{\sigma+2}\\
&+ \sum_{n=-\infty}^{\infty}(C_nC_{n+1}^\star + C_n^\star C_{n+1}) + V_0(C_0C_1^\star + C_1C_0^\star)
\end{aligned} \tag{5}$$

where $V_0 = (\frac{1}{\sqrt{K}} - 1)$ and $V = \frac{1}{\sqrt{K}}$.

3 A dimeric nonlinear impurity in the Cayley tree

Here we are interested in the possible solutions for SL states due to a dimeric impurity. Hence, we assume that

$$C_m = \phi_m e^{-iEt},$$

where

$$\phi_m = (sgn(E)\eta)^{m-1}\phi_1 \quad \text{for } m \geq 1$$

and

$$\phi_{-|m|} = (sgn(E)\eta)^{|m|}\phi_0 \quad \text{for } m \leq 0. \tag{6}$$

Equation (6) is the exact form of ϕ_m in the presence of a dimeric impurity and can be derived from Green's function analysis. $\eta \in [0,1]$ is given by $\eta = \left(|E| - \sqrt{E^2-4}\right)/2$. Sgn$(E)$ denotes the signature of E. We further define $\beta = \phi_1/\phi_0$ if $|\phi_1| \leq |\phi_0|$. Otherwise we invert the definition of β. Because of the symmetry in the system we will obtain the same result. So, apparently $\beta \in [-1,1]$. However, we shall show later that for $\chi > 0$, negative values of β except $\beta = -1$ are not permissible. The analytical argument showing the impossibility of such a situation is presented in reference [34]. Of course, negative values of $\beta \geq -1$ will produce SL states in the antisymmetric set if $\chi < 0$. So, for $\chi > 0$, $\beta \in [0,1]$ and for $\chi < 0$, $\beta \in [-1,0]$. This assertion will also be substantiated here in due course. Now, from the normalization condition we get

$$|\phi_0|^2 = \frac{1-\eta^2}{1+\beta^2}\,. \tag{7}$$

Introducing equations (6) and (7), and the definition of β in the Hamiltonian (Eq. (5)) we get an effective Hamiltonian, H_{eff} where

$$\begin{aligned}
H_{eff} =& 2sgn(E)\eta + 2\frac{V\beta(1-\eta^2)}{1+\beta^2}\\
&+ 2\frac{\chi}{\sigma+2}\frac{(1-\eta^2)^{\sigma/2+1}(1+|\beta|^{\sigma+2})}{(1+\beta^2)^{\sigma/2+1}}
\end{aligned} \tag{8}$$

and $V = \frac{1}{\sqrt{K}}$ for the Cayley tree otherwise it is a parameter. The Hamiltonian consists of two variables, namely, β and η and two constants, namely, χ and σ. The stationary localized states correspond to fixed points of the reduced dynamical system described by H_{eff}.

3.1 $|\beta| = 1$

We here consider the case where $|\beta| = 1$. From $\partial H_{eff}/\partial\beta = 0$, it is easy to see that $|\beta| = 1$ is always a solution to the equation. This is due to the symmetry in the system. For $\chi > 0, \beta = 1$ yields the symmetric set, while for $\beta = -1$ we get the antisymmetric set. In this limit the relevant equation governing the formation of SL states is obtained by setting $dH_{eff}/d\eta = 0$. This in turn yields

$$\frac{1}{\chi} = 2^{-\sigma/2}\frac{\eta(1-\eta^2)^{\sigma/2}}{sgn(E) - sgn(\beta)V\eta} = F(\sigma,\eta). \tag{9}$$

Case 1: $\sigma = 0$

Here we have a linear dimeric impurity in a Cayley tree. Since we are considering $\chi > 0$, for the symmetric case we must have $sgn(E) = 1$. On the other hand for antisymmetric case $sgn(E)$ can take both the signs. If $V < 1$, for the symmetric case $F(0,\eta)$ has a divergence at $\eta = \frac{1}{V} > 1$. Since $\eta \in [0,1]$, this divergence occurs beyond the permissible range of η. However, in the permissible range of η, $\frac{dF}{d\eta} = (1 - V\eta)^{-2} > 0$. So, $F(0,\eta)$ is a monotonically increasing function of η and it assumes the permissible maximum value at $\eta = 1$. This in turn then yields $\chi_{cr} = (1-V)$ and a SL state will be obtained if $\chi \geq \chi_{cr}$. For the dimer in a Cayley tree we then need $\widetilde{\chi} \geq \widetilde{\chi}_{cr} = \sqrt{K} - 1$. On the other hand for $V > 1$, the divergence at $\eta = \frac{1}{V}$ is in the permissible range of η. So, even with an infinitesimally small value of χ, we shall obtain a SL state. However, for $\chi > 0$, $\eta \in [0, \frac{1}{V}]$.

We consider now the antisymmetric case with $sgn(E) = +1$. It is easy to see that $F(0,\eta)$ is a monotonically increasing function of η. So, $F(0,\eta)$ takes the maximum possible value at $\eta = 1$. This in turn gives $\chi_{cr} = 1 + V$. V is implicitly assumed to be positive. So, we shall get a SL state for the Cayley tree if $\widetilde{\chi} \geq \widetilde{\chi}_{cr} = \sqrt{K} + 1$. For $sgn(E) = -1$, $F(0,\eta)$ will diverge at $\eta = \frac{1}{V}$. So, if $V < 1$, we shall not obtain any SL state below the band of the host system. On the other hand if $V > 1$, $F(0,\eta)$ diverges in the permissible range of η. However, we also have $F(0,1) = \frac{1}{V-1}$. Thus, $\chi_{cr} = V - 1$ and a SL state below the band will be obtained if $\chi \leq \chi_{cr} = V - 1$. We now summarize our findings on the linear dimer.
(i) $V < 1$. No SL state will be obtained if $\chi < 1-V$. There is one SL state for $(1-V) < \chi < (1+V)$. But there are two SL states if $\chi > (1+V)$ [35]. (ii) $V > 1$. If $0 < \chi \leq (V-1)$, there are two SL states. One appears above the band and the other lies below the band. If $(V-1) < \chi < (1+V)$, we have one SL state above the band. For $\chi \geq (V+1)$, we get two SL states and both appear above the band.

In passing we note the following. The Green's function for the Cayley tree can be obtained from equation (2). This is shown in Appendix A. Furthermore, equation (2) yields the known results for the stationary localized states when a linear dimer is embedded in a Cayley tree. These results confirm that equation (2) correctly describes the dynamics of a particle on a Cayley tree with no disorder either in the site energy or in the hopping.

Case II: $\sigma \neq 0$

Here we consider two cases, namely, $V < 1$ and $V > 1$ separately.

(A) $V < 1$. Since χ is taken to be positive, in equation (9) we need $sgn(E) = sgn(\beta) = +1$. Again the divergence of $F(\sigma,\eta)$ at $\eta = \frac{1}{V}$ is of no consequence. Furthermore, $F(\sigma,\eta)$ has at least one maximum at $\eta_m \in [0,1]$. In fact, $F(\sigma,\eta)$ has only one maximum. So, there will be a χ^s_{cr} so that for $\chi > \chi^s_{cr}$ we shall obtain two SL states and for $\chi < \chi^s_{cr}$, there will be no SL state. On the other hand in the antisymmetric case we have $sgn(\beta) = -1$ but $sgn(E)$ can be either $+1$ or -1. In the first case ($sgn(E) = +1$), $F(\sigma,\eta)$ has no divergence but $F(\sigma,0) = 0 = F(\sigma,1)$. So, $F(\sigma,\eta)$ has at least one (actually one) maximum at $\eta'_m \in [0,1]$. Consequently, we shall get another critical value of χ, say χ^a_{cr} so that if $\chi > \chi^a_{cr}$ we shall obtain two SL states. For $\chi < \chi^a_{cr}$ there will be no SL state. We further note that $\chi^a_{cr} > \chi^s_{cr}$. In the second case ($sgn(E) = -1$), $F(\sigma,\eta)$ diverges at $\eta = \frac{1}{V} > 1$. Furthermore, $F(\sigma,\eta)$ should be positive. Hence, $\eta \geq \frac{1}{V}$. Since allowed values of $\eta \in [0,1]$, no SL state will be obtained below the host band. So, we shall get three regions having no, two and four SL states. Furthermore, in half of the states $\eta \to 1$ as $\chi \to \infty$. So, these are unstable states. Equations for critical lines in the (χ,σ) plane separating three regions are given in Appendix B.

(B) $V > 1$. In the symmetric case $F(\sigma,\eta)$ diverges at $\eta = \frac{1}{V} \leq 1$ and $F(\sigma,0) = 0 = F(\sigma,1)$. Furthermore, $F(\sigma,\eta) \geq 0$ for $\eta < \frac{1}{V}$ and $F(\sigma,\eta) \leq 0$ for $\eta > \frac{1}{V}$. Hence, a SL state will always be obtained for $\chi > 0$. The maximum value of η the SL state can take is $\frac{1}{V}$ and this will happen if $\chi \sim 0$. In the antisymmetric case when $sgn(E) = +1$, $F(\sigma,\eta)$ has no divergence for $\eta \in [0,1]$. Since, $F(\sigma,0) = 0 = F(\sigma,1)$, $F(\sigma,\eta)$ has a maximum at $\eta''_m \in [0,1]$. So, there will be a critical value of χ such that $\chi < \chi_{cr}$ no SL state will be obtained. On the other hand for $\chi > \chi_{cr}$, we shall have two SL states and in one of these states $\eta \to 1$ as $\chi \to \infty$. So, one of the states is an unstable SL state. Equation for the critical line in (χ,σ) plane separating these two regions is also given in Appendix B. For $sgn(E) = -1$, $F(\sigma,\eta)$ again diverges at $\eta = \frac{1}{V}$. However, for $F(\sigma,\eta)$ to be positive we need $\frac{1}{V} \leq \eta \leq 1$. Since $F(\sigma,1) = 0$, we shall always get a SL state irrespective of the value of χ. Furthermore, the SL state will appear below the host band. The minimum value of η that a SL state in this case can attain is $\frac{1}{V}$. This will happen if $\chi \sim 0$ and $\eta \to 1$ as $\chi \to \infty$. So, this is an unstable SL state. It is then seen that for $V > 1$, we have two regions. The region below the critical line has two SL states and that above the critical line has four SL states. As usual, half of these states are unstable.

3.2 $|\beta| \neq 1$

We consider here the scenario, $|\beta| \neq 1$. H_{eff} (Eq. (8)) has now two dynamical variables, η and β. So, the relevant equation governing the formation of SL states is obtained by setting $\partial H_{eff}/\partial X_i = 0$, where $X_1 = \eta$ and $X_2 = \beta$. From the first condition ($\partial H_{eff}/\partial \eta = 0$) we obtain

$$sgn(E)\eta = \frac{sgn(\beta)}{V} \frac{|\beta|^{-\sigma/2} - |\beta|^{\sigma/2}}{|\beta|^{-(\sigma/2+1)} - |\beta|^{\sigma/2+1}} \cdot \qquad (10)$$

So, η is a symmetric function of β and β^{-1} as enunciated earlier. Furthermore, since $\eta \geq 0$, if $V > 0$, $sgn(E) = sgn(\beta)$. In subsequent discussion we assume that $sgn(\beta) = +1$. When $|\beta| \to 1$, from equation (10) we obtain,

$$\eta_u = \frac{1}{V} \frac{\sigma}{\sigma + 2} \cdot \qquad (11)$$

The second condition ($\partial H_{eff}/\partial\beta = 0$) yields

$$\frac{1}{\chi} = \frac{sgn(\beta)|\beta|\,(1-|\beta|^{\sigma})}{V\,(1+\beta^2)^{\sigma/2}\,(1-\beta^2)}\,\left(1-\eta^2\right)^{\sigma/2}. \qquad (12)$$

Since $(1-\eta^2)$ for $\eta \in [0,1]$ is a positive semidefinite quantity, for $V > 0$, χ and β will possess the same sign. In other words, SL states with $|\beta| \neq 1$ in the antisymmetric set are not possible. It is trivially seen that the right hand side of equation (12) is also a symmetric function of β and β^{-1}. We further note that introduction of equation (10) in equation (12) makes the right hand side of equation (12) an explicit function of σ and β. We call this function $F(\sigma,\beta)$.

Since $\sigma = 2$ is physically more relevant we consider this case in detail. In this situation we have

$$\frac{1}{\chi} = \eta(1-\eta^2) = F(2,\eta) = g(\eta). \qquad (13)$$

We note that $g(\eta)$ has one and only one maximum at $\eta_m^2 = \frac{1}{3}$ for $\eta \in [0,1]$. Furthermore, $\eta_u = (2V)^{-1}$. So, if $V < 0.5$ or $K > 4$, $\eta_u > 1$. Since $g(\eta)$ in this case is defined for $\eta \in [0,1]$, the maximum of $g(\eta)$ lies in the permissible range of η. On the other hand if $V \geq 0.5$ ($K \leq 4$), $\eta_u \leq 1$. So, $\eta \in [0,\eta_u]$. Then for η_m to stay in the allowed range of η, we need $\eta_m^2 \leq \eta_u^2$. This in turn yields $V \leq 0.8660$ or $K \geq 1.33$. So, there will be a lower critical value of χ, χ_{crl} such that $\chi < \chi_{crl}$ there will be no SL state and for $\chi > \chi_{crl}$ there can be two SL states. From equation (13) we further obtain $\chi_{crl} = \widetilde{\chi}_{crl}/\sqrt{K} = 2.5980$. Again for $V \leq 0.5$, since $\eta \in [0,1]$ and $g(0) = 0 = g(1)$, we shall obtain two SL states for $V \leq 0.5$ and $\chi > 2.5980$. In one of the states $\eta \to 1$ as $\chi \to \infty$. So, one state is an unstable state. On the other hand, $0.5 \leq V \leq 0.8660$, we get $\chi_{cru} = \frac{8V^3}{4V^2-1}$ and $\chi > \chi_{cru}$, we get only one SL state. Then, if $V \leq 0.8660$ and $\chi_{crl} < \chi \leq \chi_{cru}$ we have two SL states. In one state $\eta \to \eta_u$ as $\chi \to \infty$. So, one state is unstable. Thirdly, for $V > 0.8660$, $\eta \in [0,\eta_u]$ and $\eta_u < \eta_m$. So, $g(\eta)$ takes the maximum value at η_u and the corresponding critical value of χ is χ_{cru}. We note that for $V = 1$, $\chi_{cru} = 8/3$ [34,29] and for $V = \sqrt{2}$, $\chi_{cru} = 3.232$. For $\chi > \chi_{cru}$ we shall obtain one stable SL state.

We now consider the general σ. Substituting $\eta_u = 1$ in equation (11) we obtain $\sigma' = 2V/(1-V)$ if $V < 1$. For the Cayley tree it translates to $\sigma' = 2/(\sqrt{K}-1)$. For simplicity we break the discussion in two cases.

Case I: $\sigma \geq \sigma'$

Here $\eta_u \geq 1$. So, $\eta \in [0,1]$ but $\beta \in [0,\beta_u]$ where $\beta_u \leq 1$. Furthermore, $F(\beta_u,\sigma) = 0 = F(0,\sigma)$. So, for a given $\sigma \geq \sigma'$, $F(\beta,\sigma)$ has at least one maximum at $\beta_m \in [0,\beta_u]$. For $K = 4$ or $V = 0.5$, $\sigma' = 2$. In Figure 2 we have plotted $F(\beta,\sigma)$ for $\sigma = 2.5$. We see that there is only one maximum. Due to the maximum at β_m, in the (χ,σ) plane there will be a critical line separating the no state region from the region containing two SL states. The equation of the critical line is $\chi_{cr}^{(1)} = [F(\beta_m,\sigma)]^{-1}$ where β_m is the solution of $\partial F/\partial\beta$=0. Since in one of the states $\eta \to 1$ as $\chi \to \infty$, the two states region has one unstable state. Furthermore, as we go along the critical line $\beta \to 0$, $\eta \to 0$ and $\chi \to \infty$. This implies that in the stable state, the amplitude gets preferentially localized in one of the dimer sites as $\sigma \to \infty$. $\chi \to \infty$ because in the limit a monomer localized state is formed.

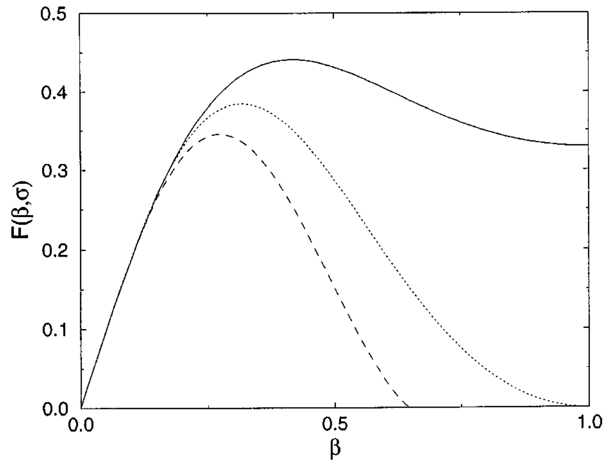

Fig. 2. $F(\beta,\sigma)$ as a function of β for $\beta \neq 1$ in case of a dimeric nonlinear impurity in a Cayley tree. Here K=4. Solid, dotted and dashes curves are for $\sigma = 1.5$, 2 and 2.5 respectively

Case II: $\sigma < \sigma'$

Since $\eta_u < 1$, $\beta \in [0,1]$. Consequently, there will be a critical value of χ, $\chi_{cr}^{(2)}$ given by

$$\chi_{cr}^{(2)} = \frac{2V}{\sigma}\left(\frac{1-\eta_u^2}{2}\right)^{-\sigma/2}. \qquad (14)$$

For $\chi > \chi_{cr}^{(2)}$, we shall obtain at least one SL state. Since $\chi_{cr}^{(2)} \to \infty$ as $\sigma \to 0$ and $\sigma \to \sigma'$ ($\eta_u \to 1$), $\chi_{cr}^{(2)}$ will assume a minimum value at σ_{min}. σ_{min} is obtained from $d\chi_{cr}^{(2)}/d\sigma$ =0. However, it gives a very complicated algebraic equation in σ. When $\sigma \to 0$, $\eta \to 0$. So, we obtain a SL state localized mostly on the dimer. Precisely for this $\chi_{cr}^{(2)} \to \infty$ as $\sigma \to 0$. Again σ increases, η increases. This will require lower values of χ. So, for $0 \leq \sigma \leq \sigma_{min}$ and $\chi \geq \chi_{cr}^{(2)}$, the system yields a SL state with $\beta \neq 1$. On the other hand, for $\sigma_{min} \leq \sigma \leq \sigma'$, albeit η increases towards unity as $\sigma \to \sigma'$, $\chi_{cr}^{(2)}$ increases towards infinity. So, the SL state obtained for $\sigma > \sigma_{min}$ (in fact σ_{cr} defined later) and $\chi \geq \chi_{cr}^{(2)}$ is unstable.

$F(\beta,\sigma)$ also develops a local maximum at $\beta_m < 1$. This is shown in Figure 2 for $K = 4$ (V=0.5) and $\sigma = 1.5$. So, there will be a critical value of σ, σ_{cr} defined as follows. If $\sigma = \sigma_{cr} + \delta$ and $\delta \to 0$, there exists an $\epsilon \to 0$ depending on δ so that $\beta_m = 1 - \epsilon$. So, for $\sigma \geq \sigma_{cr}$ there will be a lower critical value of χ, $\chi_{cr}^{(3)}$ and $\chi < \chi_{cr}^{(3)}$ no SL state will be obtained. It is trivially

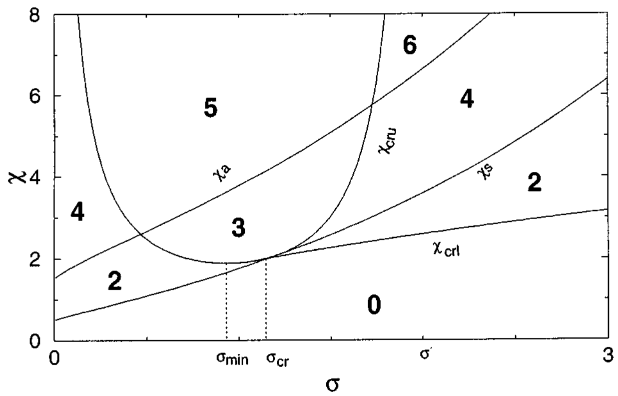

Fig. 3. Total phase diagram of SL states of a Cayley tree in presence of a dimeric nonlinear impurity. Here $K = 4(V = 0.5)$. χ^s and χ^a represents the critical lines for symmetric and antisymmetric case respectively. χ_{cru} and χ_{crl} represents the upper and lower critical line respectively for $\beta \neq 1$ case. σ_{min}, σ_{cr} and σ' are shown. Numbers indicate the number of possible SL states in those regions in the (χ, σ) plane

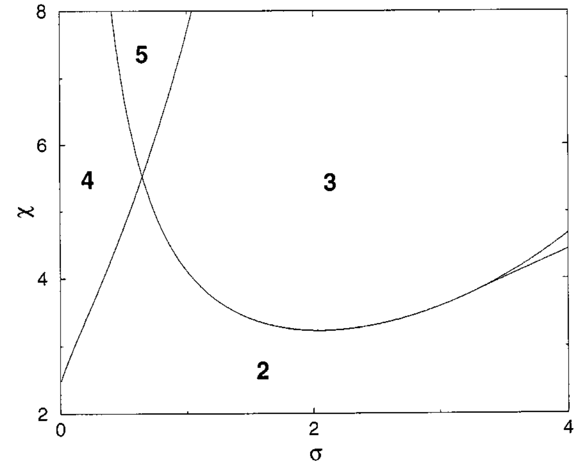

Fig. 4. Total phase diagram for SL states of a one dimensional chain with a dimeric nonlinear impurity and a bond defect in between the impurity sites. Here $V = \sqrt{2}$. Numbers indicate the number of SL states in those regions. The unnumbered closed triangular small region contains four SL states.

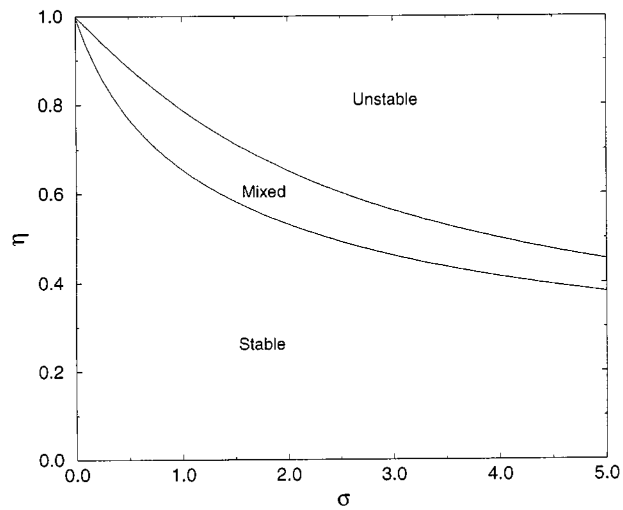

Fig. 5. Stability diagram for SL states in a Cayley tree with a dimeric nonlinear impurity. Here $K = 4$ $(V = 0.5)$. Stability of the states in various regions are marked in the figure.

seen that $\chi_{cr}^{(3)} = [F(\beta_m, \sigma)]^{-1}$. For a given $\sigma > \sigma_{cr}$ the upper critical value is $\chi_{cr}^{(2)}(\sigma)$. Then in the (χ, σ) plane we have a V-shaped region with boundaries $\sigma_{cr} \leq \sigma \leq \sigma'$ and $\chi > \chi_{cr}^{(3)}(\sigma_{cr})$ but $\chi_{cr}^{(2)}(\sigma) > \chi > \chi_{cr}^{(3)}(\sigma)$. This region contains two SL states with $\beta \neq 1$. However, one of the states is unstable because $\eta \to \eta_u$ as $\chi \to \infty$. We further note that the lower boundary of the V-shaped region joins smoothly with the critical line for $\sigma \geq \sigma'$.

Case III: $V > 1$

The scenario is very similar to the $V = 1$ case discussed in reference [34]. However σ_{cr} moves to a higher value depending on the magnitude of V.

The total phase diagram of SL states for $V = 0.5$ $(K = 4)$ is shown in Figure 3. The number of possible SL states in each region is indicated. We again note that the maximum number of SL states is six. The total phase diagram for $V = \sqrt{2}$ is shown in Figure 4. The maximum number of SL states is found to be five. Furthermore, the $\sigma = 2$ line has no special significance. The stability diagram [36] of SL states for dimeric nonlinear impurity in a Cayley tree is shown in the (η, σ) plane in Figure 5.

4 All sites nonlinear

4.1 Inter site peaked and dipped solutions

We consider here the formation of SL states in a Cayley tree with each site having a power law nonlinearity and the same coupling constant, χ. We note that the system under consideration has the required translational invariance. So, the proposed transformation (see Sect.2) will be applicable here. The problem then reduces to the study of formation of SL states in a one dimensional system with a bond defect between the zeroth and the first site. The tunneling matrix between the sites, V is reduced by a factor of $1/\sqrt{K}$ and site energies are

$$\epsilon_n = V^{\sigma(n-1)}\chi|C_n|^\sigma \qquad \text{if} \quad n \geq 1$$

and

$$\epsilon_{-|n|} = V^{\sigma|n|}\chi|C_{-|n|}|^\sigma \qquad \text{if} \quad n \geq 0. \tag{15}$$

Furthermore, the Hamiltonian is given by equation (5) with $\chi_n = \chi$, $n \in Z$. We first note that for $\chi = 0$ and $V \leq 1$, we have a band of states. For $V > 1$, we can have localized states. Again it is well known that in one dimension or in pseudo one dimension, states appearing outside the band are exponentially localized [35]. If nonlinearity can induce self-localization, states at the band edges of the system will have the propensity to undergo localization. Then low energy localized modes will have no node or all nodes and one node or $(N-1)$ nodes, depending on the sign of χ. N is the number of sites in the system. In case of states having no node, states can peak either at lattice site or at the middle of two lattice points. For the on-site peaked localized states, the monomeric ansatz is then suitable choice [27,28]. On the other hand, for the inter-site peaked or dipped states dimeric ansatz is the rational choice [20]. Since for $V=1$ and $V=1/\sqrt{K}$ we have translationally invariant one dimensional system and a Cayley tree with connectivity, K, the localized solutions will not show any space dependence. For $V>1$, we, however, do not have the required translational invariance in the system. Here localized states peaked in the vicinity of the bond defect can show the space dependence. So, for this case we restrict our study to the formation of self-localized modes pinned at the bond defect.

We first consider inter-site peaked and dipped solutions. It is clear from the previous discussion that the use of dimeric ansatz is justified for this purpose. The corresponding effective Hamiltonian with $|\beta|$=1 is given by,

$$H_{eff} = \frac{2\chi}{2^{\sigma/2}(\sigma+2)} \frac{(1-\eta^2)^{\sigma/2+1}}{(1-V^\sigma \eta^{\sigma+2})} + 2sgn(E)\eta + sgn(\beta)(1-\eta^2). \quad (16)$$

By setting $\partial H_{eff}/\partial \eta = 0$, we obtain the equation governing the formation of SL states in this system. It is given by

$$\frac{2^{\sigma/2}}{\chi} = \frac{\eta(1-\eta^2)^{\sigma/2}\,(1-(V\eta)^\sigma)}{(sgn(E)-sgn(\beta)V\eta)\,(1-\eta^2(V\eta)^\sigma)^2} = G_(\eta,\sigma). \quad (17)$$

From the asymptotic analysis of the equation of motion for $|n| \to \infty$, we obtain $\eta = (|E| - \sqrt{E^2-4})/2$ [27,28]. We further note that the effective nonlinear coupling constant in the equation of motion decreases exponentially with $|n|$ for $K>1$ and $\sigma>0$ (see Eq. (15)). We are also assuming that $\chi>0$. Since $[1-(V\eta)^\sigma] \to -\sigma\, ln(V\eta)$ as $\sigma \to 0$, $G(\eta,\sigma) \to 0$ as $\sigma \to 0$. Consequently $\chi \to \infty$. This implies that no SL state will be formed in this limit. We now consider various cases.

Symmetric case

Here $sgn(E) = sgn(\beta) = +1$. We note that $G(\eta,\sigma)$ have a removable singularity and a divergence at $\eta_0 = 1/V$ and $\eta_1 = 1/V^{\frac{\sigma}{\sigma+2}}$ respectively. But for $V<1$, $\eta_1>1$. So, the singularity of $G(\eta,\sigma)$ at η_1 will not play any role in the formation of SL states. Again $G(0,\sigma) = 0 = G(1,\sigma)$. So, $G(\eta,\sigma)$ will have at least one maximum at $\eta_m \in [0,1]$. It can be seen numerically that $G(\eta,\sigma)$ has only one maximum for $\eta \in [0,1]$. Consequently in the (χ,σ) plane there will be a critical line separating two states region from the no state region. Since one of the states in the two states region $\eta \to 1$ as $\chi \to \infty$, it is an unstable state. On the other hand for $V>1$, $G(\eta,\sigma)$ has a divergence at $\eta_1<1$. So, in this case the system will always produce a SL state even if χ is infinitesimally small. Furthermore, $\lim_{\epsilon\to 0} G(\eta_1-\epsilon,\sigma) \to \infty$ from the positive side only. So, for $\chi>0$, $\eta_{max} = \eta_1$ and $\eta \to 0$ as $\chi \to \infty$. Hence this is a stable SL state.

Antisymmetric case

In this limit $sgn(\beta) = -1$ but $sgn(E)$ can be either $+1$ or -1. We note that for $sgn(E) = +1$, $G(\eta_0,\sigma) = 0$ here. But for $V<1$ both η_0 and η_1 lie beyond unity. Since $G(\eta,\sigma) = 0$ both at $\eta = 0$ and $\eta = 1$, it has a maximum at $\eta_m \in [0,1]$. This implies that in the (χ,σ) plane there will be a critical line. This line again separates the no state region and the two states region. Furthermore, in the two state region one state will be unstable for the same argument given earlier.

For $V>1$, we note that $G(\eta,\sigma)$ goes to zero and infinity at η_0 and η_1 respectively. Furthermore, if σ is finite, $\eta_0<\eta_1$. So, for $\eta \in (\eta_0,\eta_1)$, $G(\eta,\sigma)$ is negative. This in turn implies that $\lim_{\epsilon\to 0} G(\eta_1-\epsilon,\sigma) \to -\infty$ and $\lim_{\epsilon\to 0} G(\eta_1+\epsilon,\sigma) \to \infty$. Consequently, $\lim_{\epsilon\to 0} G(1-\epsilon,\sigma) \to 0$ from the positive direction for $\eta \in [\eta_1,1]$. Therefore, we shall always obtain a SL state even if χ is infinitesimally small. In this SL state, however, $\eta_{min} = \eta_1$ and as $\eta \to 1$, $\chi \to \infty$. So, this is an unstable state. We further note that $G(0,\sigma) = 0 = G(\eta_0,\sigma)$. Then there will be a maximum of $G(\eta,\sigma)$ at $\eta_m \in [0,\eta_0]$. So, there will be a critical line in the (χ,σ) plane also. This line will separate one state region and three states region. It is further seen that two of the states in the later region are unstable. We can also have $sgn(E) = -1$. However, for $V<1$ no SL states will be obtained in this limit. On the other hand for $V>1$, $\lim_{\epsilon\to 0} G(\eta_1 \mp \epsilon,\sigma) \to \mp\infty$. So, we shall always get a SL state below the band even if χ is infinitesimally small. However, this SL state is unstable.

We now combine our results to obtain the phase diagram. For $V<1$, we have three regions, namely I, II and III containing no SL state, two SL states and four SL states respectively. This is shown in Figure 6 for $K=4$ or $V=0.5$. The stability diagram [36] of SL states for this case is shown in Figure 7. However, for $V>1$ we do not have any no SL state region. Instead we have a three state region separated from a five state region by a critical line. One of these states appears below the band. In the three state region we have two unstable states while in the other region we have three unstable states. For $V=\sqrt{2}$, the phase diagram is shown in Figure 8.

4.2 On site peaked soliton

We discuss here the formation of on-site soliton in the transformed system described by the system given by equation (2).

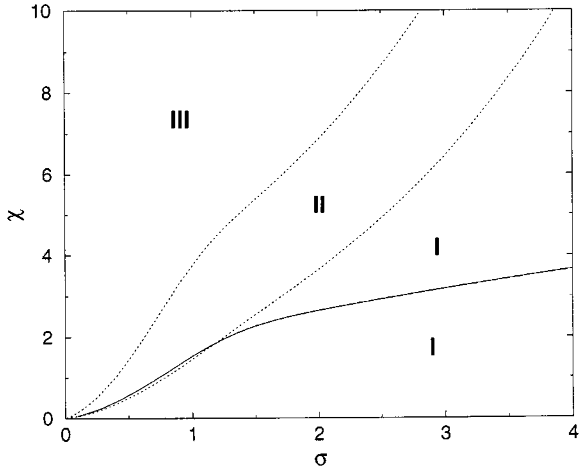

Fig. 6. Phase diagram for SL states in a fully nonlinear Cayley tree. The solid line is the critical line for the on site (zeroth site) peaked solution. The lower dotted line is the critical line for the inter site peaked (symmetric) solution and the uppermost line defines the critical line for the inter site dipped (antisymmetric) solution. Here $K = 4$ ($V = 0.5$). For inter site solutions region I, II and III contains no, two and three SL states respectively. For on site solutions there is no state below the solid curve and two states above the solid curve.

For this purpose we first consider a power law nonlinear impurity with strength, χ embedded at the zeroth site. After introducing the dimeric ansatz in the appropriate form of the Hamiltonian we obtain

$$H_{eff} = \frac{2\chi}{(\sigma+2)}\left(\frac{1-\eta^2}{1+\beta^2}\right)^{\sigma/2+1} + 2sgn(E)\ \eta + 2V\beta\left(\frac{1-\eta^2}{1+\beta^2}\right)\cdot \tag{18}$$

Again relevant equations are obtained by $\partial H_{eff}/\partial X_i=0$ where $X_1 = \beta$ and $X_2 = \eta$. After a trite algebra we then obtain $\beta = sgn(E)V\eta$ and

$$\frac{sgn(E)}{\chi} = \frac{\eta(1-\eta^2)^{\sigma/2}}{(1+V^2\eta^2)^{\sigma/2}(1-V^2\eta^2)} = f(\eta,\sigma). \tag{19}$$

For $V = 1/\sqrt{K}$, equation (19) describes the formation of SL states due to a nonlinear impurity in a Cayley tree. This has been discussed in detail in reference [34]. So, we see that the dimeric ansatz reduces to the appropriate monomeric ansatz. When $V > 1$, $f(\eta,\sigma)$ has a divergence at $\eta_u = 1/V$. Furthermore we have, $f(0,\sigma) = 0 = f(1,\sigma)$ and $\lim_{\epsilon\to 0} f(\eta_u - \epsilon,\sigma) \to \infty$. So, we shall obtain two SL states even if χ is infinitesimally small and $\sigma > 0$. However, one state will appear below the band. In this state $\eta_{min} = \eta_u$ and $\eta \to 1$ as $\chi \to \infty$. So, this is an unstable state. For, $\sigma = 0$, this state will appear if $0 < \chi < (V^2 - 1)$.

To study the formation of on-site peaked SL states in the fully nonlinear chain we put the dimeric ansatz with $\beta = sgn(E)V\eta$ in the Hamiltonian, H given by equation (5). We then obtain the effective Hamiltonian, H_{eff},

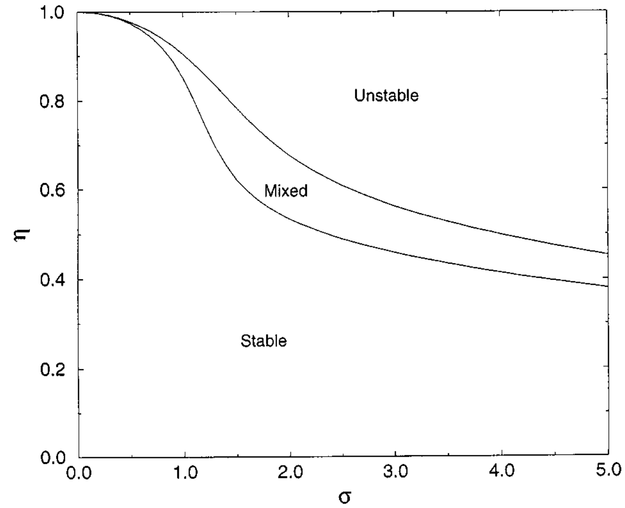

Fig. 7. Stability diagram for SL states in a fully nonlinear Cayley tree. Here $K = 4$ ($V = 0.5$). Stability of the states in various regions are marked in the figure.

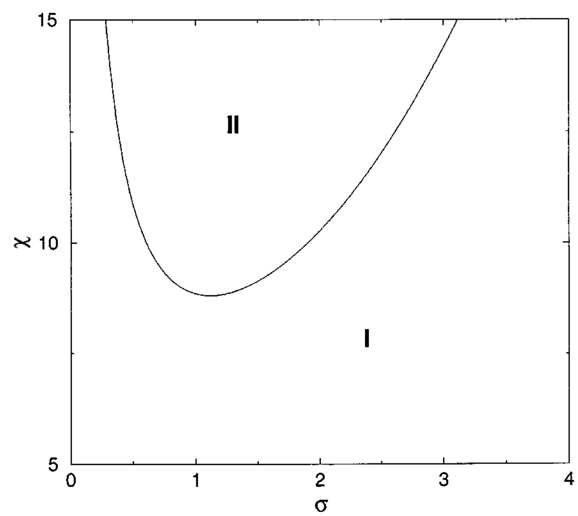

Fig. 8. Total phase diagram for SL states of a fully nonlinear one dimensional chain a dimeric nonlinear impurity and a bond defect in between the impurity sites. Here $V = \sqrt{2}$. The region I contains three SL states and the region II contains five SL states.

given by

$$H_{eff} = \frac{2\chi}{(\sigma+2)}\frac{(1-\eta^2)^{\sigma/2+1}(1+V^{\sigma+2}\eta^{\sigma+2})}{(1+V^2\eta^2)^{\sigma/2+1}(1-V^{\sigma}\eta^{\sigma+2})} + 2sgn(E)\eta + sgn(E)\frac{2V^2\eta(1-\eta^2)}{1+V^2\eta^2}\cdot \tag{20}$$

We then set $\partial H_{eff}/\partial\eta = 0$. After a trite algebra we finally obtain

$$\frac{sgn(E)}{\chi} = \frac{\eta\left(1-\eta^2\right)^{\sigma/2}\left(1+V^{\sigma+2}\eta^{\sigma+4}\right)}{\left(1+V^2\eta^2\right)^{\sigma/2}\left(1-V^{\sigma}\eta^{\sigma+2}\right)^2}\frac{\left(1-V^{\sigma}\eta^{\sigma}\right)}{\left(1-V^2\eta^2\right)} = f_1(\eta,\sigma). \tag{21}$$

When $V = 1$, equation (21) reduces to the relevant equation in reference [28]. We note that $f_1(\eta, \sigma)$ has a removable singularity at η_u and a divergence at $\eta_1 = V^{\frac{-\sigma}{\sigma+2}}$. However, for $V < 1$ the divergence at η_1 does not play any role in the formation of SL states. Since $f_1(0, \sigma) = 0 = f_1(1, \sigma)$ we expect at least one maximum of $f_1(\eta, \sigma)$ at $\eta_m \in [0, 1]$. It is seen numerically that for $\eta \in [0, 1]$ $f_1(\eta, \sigma)$ has only one maximum. So in the (χ, σ) plane there is a critical line separating the no state region from the two states region. Of course, one of the states is unstable. In Figure 6 the critical line is shown by solid curve for $K = 4$. On the other hand, for $V > 1$, $f_1(\eta, \sigma)$ diverges at η_1. Since, $\lim_{\epsilon \to 0} f_1(\eta_1 \mp \epsilon, \sigma) \to \infty$ we have also two states and these states are formed even if χ is infinitesimally small. One of the states is unstable. However, numerical calculation shows that there exists a critical value of σ say σ_{cr} such that for $\sigma > \sigma_{cr}$ there will be a four state region bounded by two critical values of χ. For example, for $V = \sqrt{2}$ $\sigma_{cr} \sim 3.85$. But the four states region occurs at larger value of χ. Two of the states are again unstable.

We now end this section with a brief discussion on the exactness of the calculation. The method adopted here is similar to the well known effective medium theory for the linear system. This is quite clear from the form of H_{eff} given in equations (16, 20) In the first case we have an effective nonlinear dimer in which χ_{eff} is a function of η, χ and σ. In the second case we have a effective nonlinear monomer. But the use of the dimeric as well as the monomeric ansatzs are quite justified for the study of low energy self-localized modes. This has been discussed. Therefore, basic features obtained here will also be reproduced by rigorous calculations. But quantitative agreement may not be obtained. More work is therefore necessary.

5 Summary

The DNLSE with general power law nonlinearity is used to study the formation of stationary localized states in the Cayley tree. Importance of this study is discussed in the introduction. Two cases, namely, a dimeric nonlinear impurity and the fully nonlinear system are considered. To facilitate the study a transformation is devised to map the system to an one dimensional system with a bond defect. We also note in passing that the problem can also be mapped to an half infinite chain with a bond defect between the zeroth site and the subsequent site. The Cayley tree Green's function for the problem can be obtained from the transformed system. This is discussed in Appendix A.

The formation of SL states is studied by analyzing the fixed point equations of the reduced dynamical system. This is obtained by introducing the dimeric ansatz in the appropriate Hamiltonian. For the linear dimer our results agree with known results. In case of a nonlinear dimer impurity, the system is found to sustain two types of SL states and altogether a maximum of *six* types of SL states can be obtained. In one case the absolute amplitude at two sites are equal. In the second category we find states with unequal amplitudes. In this aspect our results are very similar to what we obtain for a one dimensional chain. There are, however, some differences. In the one dimensional chain the no state region is obtained for $\sigma \geq 2$. Furthermore, the V region extends to infinity $(\sigma' \to \infty)$. Here, no state region is obtained from $\sigma = 0$, and $\sigma' = \frac{2}{(\sqrt{K}-1)}$. So, the V region shrinks as K increases. For $\sigma = 2$, we find that for $K > 1.33$ $(V < 0.86)$, a third critical value of χ, $\chi_{cr} = \widetilde{\chi}_{cr}/\sqrt{K} = 2.5980$. This is a very interesting result. The corresponding χ_{cr} for $K = 1$ is 8/3. The stability and the phase diagram of SL states are discussed in detail.

The formation of SL states in the fully nonlinear Cayley tree is also considered. For the on site peaked solution, the appropriate ansatz is derived. In the perfect nonlinear chain, a three SL states region exists for the on site peaked SL state. For the Cayley tree, this region is absent. Instead for all cases, we find a two states region and a no state region separated by respective critical lines. Along with this the case where V (in relation to other hopping element) > 1 is considered. For this case we show that under certain conditions, states can appear both below and above the band. Furthermore, we also find a four states region.

The nonlinear dimer is considered here. But to obtain a good understanding of SL states in fully nonlinear lattices, a systematic study of SL states due to nonlinear clusters of various sizes embedded in a perfect lattice is necessary. So, the next thing to consider is cluster of three and four nonlinear sites in a perfect system. This work is in progress.

Appendix A

We show here that a subset of amplitude Green's functions of equation (4) with $\epsilon_n = 0$, $n \in Z$ yields the amplitude Green's functions of a particle moving on a Cayley tree with connectivity, K. We first note that $G_{n,m}(\widetilde{E} = E\sqrt{K}) = \frac{1}{\sqrt{K}} \widetilde{C}_m(0) G_{n,m}(E)$. Furthermore, from the transformations, we have $\widetilde{C}_m(0) = C_m(0)/K^{(m-1)/2}$ if $m \geq 1$ and $\widetilde{C}_m(0) = C_m(0)/K^{|m|/2}$ if $m \leq 0$. The Hamiltonian (H) that yields equation (4) is $H = H_0 + H_1$ where,

$$H_0 = \sum_n \left(a_n^\dagger a_{n+1} + a_{n+1}^\dagger a_n \right)$$

and

$$H_1 = \left(\frac{1}{\sqrt{K}} - 1) \right) \left(a_0^\dagger a_1 + a_1^\dagger a_0 \right) \tag{22}$$

$a_n (a_n^\dagger)$ destroys (creates) a particle at the n-th site. We define $V_0 = (1/\sqrt{K} - 1)$. We further have $G(E) = (E - H)^{-1}$ and $G_0(E) = (E - H_0)^{-1}$. These two operators are related by $(I - G_0 H_1) G = G_0$ where I is the identity operator. By defining $< n|G(E)|m >= G_{n,m}(E)$ and with similar definition for $G_{0(n,m)}(E)$ we obtain from the relation between G and G_0

$$G_{n,m}(E)+V_0\left(G_{0(n,1)}(E)G_{0,m}(E)+G_{0(n,0)}(E)G_{1,m}(E)\right) = G_{0(n,m)}(E). \quad (23)$$

We have then two unknowns, namely, $G_{0,m}(E)$ and $G_{1,m}(E)$. After some algebra we obtain

$$\begin{pmatrix} G_{0,m}(E) \\ G_{1,m}(E) \end{pmatrix} = \frac{1}{D}\begin{pmatrix} 1-V_0G_{0(1,0)}(E) & V_0G_{0(0,0)}(E) \\ V_0G_{0(0,0)}(E) & 1-V_0G_{0(0,1)}(E) \end{pmatrix} \times \begin{pmatrix} G_{0(0,m)}(E) \\ G_{0(1,m)}(E) \end{pmatrix} \quad (24)$$

and

$$D=\left(1-V_0G_{0(0,1)}(E)\right)\left(1-V_0G_{0(1,0)}(E)\right)-V_0^2G_{0(1,0)}^2(E). \quad (25)$$

We again note that $G_{0(m,n)}(E) = [sgn(E)]^{n-m+1} \times G_{0(0,0)}(|E|)\eta^{|n-m|}$ and $G_{0(0,0)}(|E|) = 1/\sqrt{E^2-4}$ for $|E|>2$. Hence, $G_{0(0,0)}(|E|) = (1-\eta^2)/\eta$. After some simple algebra we obtain $D = [(K-1)\eta G_{0(0,0)}(|E|)+K]/K$. Furthermore, from equation (24) and relevant transformations we obtain

$$\widetilde{G}_{0,0}(\widetilde{E}) = \frac{1}{\sqrt{K}}G_{0,0}(E) = sgn(\widetilde{E})\frac{2K}{(K-1)|\widetilde{E}|+(K+1)\sqrt{\widetilde{E}^2-4K}}\cdot \quad (26)$$

Again from equation (24) we can easily show that for $m>0$

$$\widetilde{G}_{o,m}(\widetilde{E}) = \frac{1}{\sqrt{K}}\frac{1}{K^{(m-1)/2}}G_{0,m}(E) = [sgn(\widetilde{E})]^{m+1}\widetilde{G}_{0,0}(|\widetilde{E}|)\left(\frac{2}{|\widetilde{E}|+\sqrt{\widetilde{E}^2-4K}}\right)^{|m|} \quad (27)$$

when $\widetilde{G}_{0,0}(|\widetilde{E}|)$ is given by equation (26). On the other hand for $m<0$, we have

$$\widetilde{G}_{0,-|m|}(\widetilde{E}) = \frac{1}{\sqrt{K}}\frac{1}{K^{|m|/2}}G_{0,-|m|}(E) = \widetilde{G}_{0,m}(\widetilde{E}). \quad (28)$$

From equation (24) we further obtain, for $m \geq 1$

$$\widetilde{G}_{1,m}(\widetilde{E}) = \frac{1}{\sqrt{K}}\frac{1}{K^{(m-1)/2}}G_{1,m}(E) = [sgn(\widetilde{E})]^{m}\widetilde{G}_{0,0}(\widetilde{E})\left(\frac{2}{|\widetilde{E}|+\sqrt{\widetilde{E}^2-4K}}\right)^{m-1} = \widetilde{G}_{0,m-1}(\widetilde{E}). \quad (29)$$

On the other hand for $m \leq 0$ we obtain

$$\widetilde{G}_{1,-|m|}(\widetilde{E}) = \frac{1}{\sqrt{K}}\frac{1}{K^{|m|/2}}G_{1,-|m|}(E) = \widetilde{G}_{0,|m|+1}(\widetilde{E}). \quad (30)$$

Since in relation to the Cayley tree we are dealing with a translationally invariant problem, the choice of origin is arbitrary. So, (0,1) bond can be rechristened $(n,n+1)$ without any loss of generality. If the shift in the origin is incorporated in the transformation, $G_{0,m}(E)$ and $G_{1,m}(E)$ will be transformed to $\widetilde{G}_{n,m}(\widetilde{E})$ and $\widetilde{G}_{n+1,m}(\widetilde{E})$ respectively. Consequently our result will agree in full with the calculation in reference [35].

Appendix B

We derive here the equation of the critical line for the symmetric state with β=1 and for the antisymmetric state. Note that $\chi>0$ and $V>0$. The equation to be considered for the purpose is

$$\frac{2^{\sigma/2}}{\chi} = \frac{\eta(1-\eta^2)^{\sigma/2}}{sgn(E)-sgn(\beta)V\eta} = F(\eta,\sigma). \quad (31)$$

Then the equation of the critical line in (χ,σ) plane is

$$\chi_{cr}^{\pm} = \frac{2^{\sigma/2}}{F(\eta_{cr}^{\pm},\sigma)}\cdot \quad (32)$$

In equation (32) $\pm$ refers to the symmetric and the antisymmetric cases respectively. To find $\eta_{cr}^{\pm}$ we set $\partial F/\partial\eta = 0$. This in turn yields

$$\sigma V\eta^3 \mp (\sigma+1)\eta^2 \pm 1 = 0. \quad (33)$$

In equation (33) the upper sign refers to the symmetric case and the lower sign to the antisymmetric case. For $V=1$, we find that

$$\sigma\eta^2 \mp \eta - 1 = 0. \quad (34)$$

Equation (34) has been derived in reference [34]. η_{cr} is a real positive root of equation (33) and it must be less than unity. The expression for the η_{cr} is found to be

$$\eta_{cr}^{\pm} = \frac{\sigma+1}{3\sigma V} \pm Im\left(\frac{\sqrt{3}}{6\sigma V}\left(\frac{Q}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}\,(\sigma+1)^2}{Q}\right)\right) \mp \frac{1}{6\sigma V}\left(\frac{\sqrt[3]{2}\,(\sigma+1)^2}{Q} + \frac{Q}{\sqrt[3]{2}}\right) \quad (35)$$

where, $Q = \sqrt[3]{(A+3\sigma V\sqrt{3B})}$, $A = 2+6\sigma+6\sigma^2+2\sigma^3 - 27\sigma^2V^2$ and $B = -4-12\sigma-12\sigma^2-4\sigma^3+27\sigma^2V^2$. In equation (35) Im refers to the imaginary part. While the upper sign refers to the symmetric case, lower sign is for the antisymmetric case.

References

1. Yu.S. Kivshar, B.A. Malomed, Rev. Mod. Phys. **61**, 763 (1993).

2. T. Holstein, Ann. Phys. **8**, 325 (1959); L.A. Turkevich, T.D. Holstein, Phys. Rev. B **35**, 7474 (1987).
3. J.C. Eilbeck, P.S. Lomdahl, A.C. Scott, Physica **D16**, 318 (1985).
4. *Davydov's Soliton Revisited: Self-Trapping of Vibrational Energy in Protein,* Vol. 243 of *NATO ASI Series, Series B: Physics,* edited by P.L. Christiansen *et al.* (Plenum, New York, 1991).
5. D. Chen, M.I. Molina, G.P. Tsironis, J. Phys. Condens. Matter **5**, 8689 (1993).
6. P.L. Christiansen *et al.*, Phys. Rev. E **54**, 924 (1996).
7. P.L. Christiansen *et al.*, Phys. Rev. B **55**, 5759 (1997).
8. Yi Wan, C.M. Soukoulis, Phys. Rev. B **40**, 12264 (1989).
9. M.I. Molina, G.P. Tsironis, Phys. Rev. Lett. **73**, 464 (1994).
10. V.M. Kenkre, D.K. Campbell, Phys. Rev. B **34**, 4959 (1986).
11. V.M. Kenkre, G.P. Tsironis, D.K. Campbell, Nonlinearity in Condensed Matter., edited by A.R. Bishop *et al.* (Springer-Verlag, 1987).
12. D.H. Dunlap, V.M. Kenkre, P. Reinekar, Phys. Rev. B **47**, 14842 (1992).
13. P.K. Datta, K. Kundu, Phys. Rev B **53**, 14929 (1996).
14. D. Henning, B. Esser, Phys. Rev A **46**, 4569 (1992).
15. B.C. Gupta, K. Kundu, Indian J. Phys. **70A(6)**, 747 (1996).
16. M. Johansson, R. Riklund, Phys. Rev. B **49**, 6587 (1994).
17. M.H. Cohen, E.N. Economou, C.M. Soukoulis, Phys. Rev. Lett. **51**, 1202 (1983).
18. W. Chen, D.L. Mills, Phys. Rev. Lett. **58**, 160 (1987).
19. A.A. Abrikosov, A.I. Buzdin, M.L. Kulic, Supercond. Sci. Technol. **1**, 260 (1989).
20. A.B. Aceves *et al.*, Phys. Rev. E **53**, 1172 (1996).
21. V.E. Zakharov, A.B. Shabao, Sov. Phys. JETP **61**, 118 (1971).
22. A.A. Vakhnenko, Yu.B. Gaididei, Theo. Math. Fiz. **68**, 350 (1986) [Theor. Math. Phys. **68**, 873 (1987)].
23. Ch. Claude, Yu.S. Kivshar, O. Kluth, K.H. Spatschek, Phys. Rev. B **47**, 14228 (1993).
24. A.B. Ablowitz, J.F. Ladik, J. Math. Phys. **17**, 1011 (1976).
25. T. Dauxois, M. Peyrard, A.R. Bishop, Phys. Rev. E **47**, 684 (1993).
26. D. Cai, A.R. Bishop, N. Gronbech-Jensen, Phys. Rev. Lett. **72**, 591 (1994).
27. B. Melomed, M.I. Weinstein, Phys. Lett A **220**, 91 (1996).
28. B.C. Gupta, K. Kundu, Phys. Lett. A **235** 176 (1997).
29. B.C. Gupta, K. Kundu, Phys. Rev. B **55**, 11033 (1997).
30. M.I. Molina, G.P. Tsironis, Int. Jour. mod. Phys. B **9**, 1899 (1995).
31. G.P. Tsironis, M.I. Molina, D. Hennig, Phys. Rev. E **50**, 2365 (1994).
32. Y.Y. Yiu, K.M. Ng, P.M. Hui, Phys. Lett. A **200**, 325 (1995); Solid State Commun. **95**, 801 (1995).
33. P.M. Hui, Y.F. Woo, W. Deng, J. Condens. Matter **8**, 2011 (1996).
34. B.C. Gupta, K. Kundu, Phys. Rev. B **55**, 894 (1997).
35. E.N. Economou, *Green's Function in Quantum Physics*, (Springer-Verlag Berlin, 1983).
36. E.W. Laedke, K.H. Spatschek, S.K. Turitsyn, Phys. Rev. Lett. **73**, 1055 (1994).